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THE GENERATING FUNCTION FOR THE SOLUTION OF ODEs AND ITS DISCRETE METHODS

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Abstract—In this paper, we mainly study the generating function of ODEs and its series expansion. We extend the time-centered Euler scheme with second order accuracy to an arbitrary mth order accuracy. This scheme is L-stable. The conservation laws of this scheme have been studied. This extension also provides a means to study the approximation of noncanonical Hamiltonian systems. It can also find wider applications in time integration procedures for transient analysis in mechanics and the semidiscretization of systems of PDEs.

1. INTRODUCTION

In the study of numerical methods for the solution of differential equations expressed as the Hamiltonian formalism in classical mechanics, the most important of all the symplectic schemes for Hamiltonian systems is the time-centered symplectic scheme, which preserves all the linear and quadratic conservative quantities [1–7] of Hamiltonian systems, and is stable for linear stable Hamiltonian systems [8]. This scheme is a generalization of the time-centered Euler scheme with second order accuracy to an arbitrary mth order accuracy for Hamiltonian systems. In this paper, we generalize the results of Refs [1, 3] for ODEs. The theory of the generating function may provide new formulas for the expansion of the solution of ODEs and new schemes for its numerical methods. It can also find wider applications in time integration procedures for transient analysis in mechanics and semidiscrete systems of PDEs. These schemes are one-step methods.

During recent years, the dynamical properties of one-step methods of ODEs have been studied extensively [7–10]. We are mainly interested in the conservation laws of one-step methods. Most one-step methods preserve the linear conservation laws of analytic ODEs, but for nonlinear ODEs special treatment is required. A number of authors [1, 2, 6, 11] have proposed schemes and other methods conserving the Hamiltonian energy and Professor Feng's time-centered symplectic schemes preserve all first integrals of Hamiltonian systems for linear Hamiltonian systems. Schemes preserving the Hamiltonian energy for nonlinear Hamiltonian systems have also been considered. Gear [9] presented methods maintaining the equality and inequality constraints for the numerical computation of the solution of ODEs. Greenspan [12] proposed the theory of discrete mechanics, preserving total energy and momentum for particle dynamics (see also Ref. [13] for the methods of discrete mechanics of arbitrary order). The energy preserved by sympletic methods can be formally constructed [14]. Also, a variable-step scheme which can preserve the conservation laws of systems is under consideration presently [e.g. 15].

The time-centered Euler scheme preserves not only linear but also quadratic conservative quantities for general autonomous systems. Therefore it is necessary to generalize this scheme to an arbitrary mth order accuracy, which also preserves these properties. The generalization of this scheme has been done for Hamiltonian systems [1, 2, 7], and in Ref. [7] the author proved that time-centered sympletic schemes can preserve the linear and quadratic energy for general autonomous Hamiltonian systems. Obviously, these schemes are suitable for autonomous analytic ODEs also.

To date, the time-differencing procedure of semidiscrete systems of PDEs has been commonly used with the Taylor series expansion. It is hoped that this procedure can be used extensively in the numerical integration of semidiscrete systems of PDEs.

2. THE GENERATING FUNCTION FOR THE SOLUTION OF ODEs

Let us consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) \tag{1}$$

where F: $\mathbb{R}^N \to \mathbb{R}^N$ is an analytic function in $\mathbb{R}^N$,

$$F = (F_1, F_2, \ldots, F_N)^{\mathrm{T}}$$

and study the generating function of its solution. For time-dependent ODEs, we can use a new time variable τ, let $t = \tau$ and

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1$$

then the time-dependent ODEs in $\mathbb{R}^N$ can be rewritten as time-independent equations in $\mathbb{R}^{N+1}$. For clarity, we consider only the time-independent case.

We match another system

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -(X^{\mathrm{T}}F'(x))^{\mathrm{T}} \tag{2}$$

to system (1) to make the Hamiltonian of system (1), where

$$F'(x) = \frac{\mathrm{D}F}{\mathrm{D}x}.$$

The Hamiltonian energy is $H = X^{\mathrm{T}}F(x)$. Hence, systems (1) and (2) can be rewritten as a Hamiltonian system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial X}$$

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\frac{\partial H}{\partial x} \tag{3}$$

and the flow of system (3) can be generated by the generating function of Hamiltonian system (3) as in Ref. [1]. The following theorem is the revised form of the theorems in Ref. [1], which appeared in general form in Ref. [7] for a general Hamiltonian system.

Theorem 1

Let

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4}$$

be a nondegenerate $2N \times 2N$ matrix, such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & J_{2N} \\ J_{2N} & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} J_{2N} & 0 \\ 0 & -J_{2N} \end{pmatrix} \tag{5}$$

and $C + D$ is an $N \times N$ nonsingular matrix, where

$$J_{2N} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \quad \text{and} \quad I_N = \mathrm{diag}(1, 1, \ldots, 1).$$

Denote by

$$Z = \begin{pmatrix} x \\ X \end{pmatrix}, \qquad \bar{Z} = \begin{pmatrix} \bar{x} \\ \bar{X} \end{pmatrix}$$

a time-dependent flow of system (3), which is a J_{2N}-canonical transformation, namely

$$M^{\mathrm{T}} J_{2N} M = J_{2N}$$
$$M = \frac{\partial \bar{Z}}{\partial Z} \tag{6}$$

where $\bar{Z}|_{t=0} = Z = g(Z, 0)$.

Let

$$\begin{pmatrix} \bar{W} \\ W \end{pmatrix} = T \begin{pmatrix} \bar{Z} \\ Z \end{pmatrix} \tag{7}$$

then there exists, for sufficiently small t, a time-dependent transformation $W \to \bar{W} = f(W, t)$, with Jacobian

$$f_W(W, t) \in S_m(2N)$$

namely, a $2N \times 2N$ symmetric matrix, and there also exists a generating function $\phi(W, t)$, such that

$$[\bar{W} - f(W, t)]_{\bar{W} = A\bar{Z} + BZ,\, W = C\bar{Z} + DZ} = 0 \tag{8}$$

which is the implicit representation of the J_{2N}-canonical transformation $\bar{Z} = g(Z, t)$ and

$$J\phi_W(W, t) = f(W, t)$$
$$\phi_t(W, t) = H(A_1 \phi_W(W, t) + B_1 W) \tag{9}$$

where

$$C_1 = JD^{\mathrm{T}}J, \quad D_1 = JB^{\mathrm{T}}J, \quad A_1 = -JC^{\mathrm{T}}J, \quad B_1 = -JA^{\mathrm{T}}J.$$

If $H(Z)$ depends analytically on Z, then $\phi(W, t)$ can be expressed as the convergent power series in t for sufficiently small t:

$$\phi(W, t) = \sum_{K=0}^{\infty} \phi^{(K)}(W) t^K \tag{10}$$

where

$$\phi^{(0)} = \tfrac{1}{2} W^{\mathrm{T}} N_0 W$$
$$N_0 = J^{-1}(A + B)(C + D)^{-1} \quad \text{(a symmetric matrix)}$$
$$\phi^{(1)} = H(E_0 W)$$
$$E_0 = (C + D)^{-1}$$
$$\phi^{(K+1)} = \frac{1}{K+1} \sum_{m=1}^{K} \frac{1}{m!} \sum_{i_1, \ldots, i_m}^{1, 2N} H_{z_{i_1}, z_{i_2}, \ldots, z_{i_m}}(E_0 W)$$
$$\times \sum_{\substack{K_1 + \cdots + K_m = K \\ K_j \geq 1}} (A_1 \phi_W^{(K_1)}(W))_{i_1}, \ldots, (A_1 \phi_W^{(K_m)}(W))_{i_m}, \quad K = 1, 2, \ldots. \tag{11}$$

Q.E.D.

Let $H = X^{\mathrm{T}} F(x)$, then the solution of system (3) can be generated by the generating function $\phi(W, t)$:

$$A\bar{Z} + BZ = J\phi_W(W, t)|_{W = C\bar{Z} + BZ}$$
$$\phi_t(W, t) = H(A_1 J\phi_W(W, t) + B_1 W). \tag{12}$$

Actually, equations (12) are general formulas for the generating function of the solution of system (1) if we let $X(0)=0$, so that $X(t)=0$, $\forall 0 \leqslant t \leqslant \infty$, and then formulas (12) do not depend on X. Now we consider some special cases for the generating function of the solution of system (1).

The work of Feng [1] provides three types of generating function for the solution of Hamiltonian systems, which correspond to three transformation matrices T. We now give the corresponding results for the generating function of the solution of autonomous ODEs.

Type A

$$T_{2N} = \begin{bmatrix} 0 & -I_N & 0 & 0 \\ 0 & 0 & I_N & 0 \\ 0 & 0 & 0 & I_N \\ I_N & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

$$\bar{W} = \begin{pmatrix} -\bar{X} \\ x \end{pmatrix}, \qquad W = \begin{pmatrix} X \\ \bar{x} \end{pmatrix} \tag{14}$$

and

$$N_0 = J_{2N}^{-1}(A+B)(C+D)^{-1} = -\begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}. \tag{15}$$

Now, equations (12) can be rewritten as

$$-\bar{X} = -X + \sum_{K=1}^{+\infty} \frac{\partial \phi^{(K)}}{\partial \bar{x}} (\bar{x}, X) t^K \tag{16a}$$

and

$$x = \bar{x} - \sum_{K=1}^{+\infty} \frac{\partial \phi^{(K)}}{\partial X} (\bar{x}, X) t^K \tag{16b}$$

where ϕ is the solution of the following equation:

$$\phi_t(\bar{x}, X, t) = \left(\frac{\partial \phi}{\partial \bar{x}}\right)^{\mathrm{T}} F(\bar{x}). \tag{17}$$

We differentiate equation (17) with respect to X,

$$(\phi_X)_t(\bar{x}, X, t) = \frac{\partial}{\partial \bar{x}} (\phi_X)^{\mathrm{T}} F(\bar{x}) \tag{18}$$

and express ϕ_X as a power series of t at $(\bar{x}, X)$:

$$\phi_X = \sum_{i=0}^{+\infty} \frac{t^i}{i!} \psi_i. \tag{19}$$

We then substitute equation (19) into equation (18), hence we obtain

$$\begin{aligned} \psi_0 &= \bar{x} \\ \psi_1 &= -\left(\frac{\partial}{\partial \bar{x}} \psi_0\right)^{\mathrm{T}} F(\bar{x}) \\ \psi_{i+1} &= -\left(\frac{\partial}{\partial \bar{x}} \psi_i\right)^{\mathrm{T}} F(\bar{x}), \quad i = 1, 2, \ldots. \end{aligned} \tag{20}$$

We rewrite equation (16b) as

$$x = \bar{x} + \sum_{K=1}^{+\infty} \psi_K(\bar{x}, X) t^K / K!. \tag{21}$$

Because the solution of system (1) does not depend on X,

$$\psi_K(\bar{x}, X) = \psi_K(\bar{x}), \quad K = 1, 2, \ldots \tag{22}$$

this scheme is the implicit Taylor series of the solution of system (1) in terms of the derivatives of the function $F(\bar{x})$.

Type B

$$T_{2N} = \begin{bmatrix} -I_N & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_N \\ 0 & 0 & I_N & 0 \\ 0 & -I_N & 0 & 0 \end{bmatrix} \tag{23}$$

$$\bar{W} = -\begin{pmatrix} \bar{x} \\ X \end{pmatrix}, \qquad W = \begin{pmatrix} x \\ -\bar{X} \end{pmatrix} \tag{24}$$

and

$$N_0 = -\begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}, \qquad E_0 = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}. \tag{25}$$

Now equations (12) can be rewritten as

$$\bar{x} = x + \sum_{K=1}^{+\infty} \frac{\partial \phi^{(K)}}{\partial \bar{X}}(x, \bar{X}) t^K \tag{26a}$$

and

$$X = \bar{X} + \sum_{K=1}^{+\infty} \frac{\partial \phi^{(K)}}{\partial x}(x, \bar{X}) t^K \tag{26b}$$

where

$$\phi_t(x, \bar{X}, t) = \bar{X}^{\mathrm{T}} F\left(\frac{\partial \phi}{\partial \bar{X}}\right) \tag{27a}$$

and

$$\phi = \sum_{K=0}^{+\infty} \phi^{(K)}(x, \bar{X}) t^K. \tag{27b}$$

It is easy to prove that $\phi_{\bar{X}}$ does not depend on $\bar{X}$, therefore, we differentiate equations (27a, b) with respect to $\bar{X}$:

$$(\phi_{\bar{X}})_t = F\left(\frac{\partial \phi}{\partial \bar{X}}\right). \tag{28}$$

Let

$$\psi = \phi_{\bar{X}} = \sum_{i=0}^{+\infty} \frac{t^i}{i!} \psi_i \tag{29}$$

we obtain

$$\psi_t = F(\psi) \tag{30}$$

and

$$\begin{gathered} \psi_0 = x \\ \psi_{i+1} = \left(\frac{\partial}{\partial x} \psi_i\right)^{\mathrm{T}} F(x) \\ i = 0, 1, 2, \ldots. \end{gathered} \tag{31}$$

This formula is the explicit Taylor series of the solution of system (1) at x.

Type C

$$T_{2N} = \begin{pmatrix} I_{2N} & -I_{2N} \\ \frac{1}{2}I_{2N} & \frac{1}{2}I_{2N} \end{pmatrix} \tag{32}$$

$$W = \tfrac{1}{2}(Z + \bar{Z}), \qquad \bar{W} = (\bar{Z} - Z) \tag{33}$$

and

$$N_0 = 0, \qquad E_0 = I_{2N}. \tag{34}$$

Now equations (12) can be rewritten as

$$\bar{x} - x = \sum_{K=1}^{+\infty} \phi^{(K)}_{1/2(X+\bar{X})}(\tfrac{1}{2}(x + \bar{x}), \tfrac{1}{2}(X + \bar{X}))t^K \tag{35a}$$

and

$$\bar{X} - X = -\sum_{K=1}^{+\infty} \phi^{(K)}_{1/2(x+\bar{x})}(\tfrac{1}{2}(x - \bar{x}), \tfrac{1}{2}(X + \bar{X}))t^K \tag{35b}$$

where

$$\phi_t(x^*, X^*, t) = (X^* - \phi_{x^*})^{\mathrm{T}} F(x^* + \phi_{X^*}) \tag{36}$$

and

$$\phi = \sum_{K=1}^{+\infty} \phi^{(K)}(x^*, X^*)t^K, \qquad X^* = \tfrac{1}{2}(X + \bar{X}), \qquad x^* = \tfrac{1}{2}(x + \bar{x}). \tag{37}$$

It is easily known that $\phi_{1/2(X+\bar{X})}$ does not depend on $\frac{1}{2}(\bar{X} + X)$, therefore, we differentiate equation (36) with respect to $X^* = \frac{1}{2}(X + \bar{X})$:

$$(\phi_{X^*})_t = (I - (\phi_{X^*})_{x^*})^{\mathrm{T}} F(x^* + \phi_{X^*}). \tag{38}$$

Let

$$\psi = \phi_{X^*} = \sum_{K=1}^{+\infty} \frac{t^K}{K!}\psi_K \tag{39}$$

hence

$$\psi_t = (I - \psi_{x^*})^{\mathrm{T}} F(x^* + \psi) \tag{40}$$

and

$$\psi_K(x) = \frac{\partial^K}{\partial t^K}\psi\,|_{t=0}, \quad K = 1, 2, \ldots. \tag{41}$$

Because equation (39) is symmetric with respect to x and $\bar{x}$, we can obtain

$$\psi_{2i} = 0, \quad \text{for } i = 1, 2, \ldots.$$

The first three terms of expansion (39) are given by

$$\psi_1(x^*) = F(x^*)$$

$$\psi_2(x^*) = 0$$

$$(\psi_3(x^*))_i = \frac{1}{4}\sum_{j=1}^{N}\sum_{k=1}^{N} (F_{ix_jx_k}F_jF_k - 2F_{jx_k}F_jF_{ix_k})(x^*), \quad i = 1, 2, \ldots, N \tag{42}$$

where

$$F_{jx_k} = \frac{\partial}{\partial x_k} F_j(x) \tag{43a}$$

$$F_{ix_jx_k} = \frac{\partial^2}{\partial x_j \partial x_k} F_i(x), \quad j = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, N, \quad k = 1, 2, \ldots, N \tag{43b}$$

Therefore, we obtain the series expansion of the solution of system (1) time-centered point $\frac{1}{2}(\bar{x}+x)$:

$$\bar{x}-x=\sum_{K=1}^{+\infty}\frac{t^K}{K!}\psi_K(\tfrac{1}{2}(\bar{x}+x)). \tag{44}$$

Now we give the recursive formulas of expression (39) for the scalar equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=F(x)\in\mathbb{R}^1.$$

Let

$$\Psi_m=\frac{\partial^m\psi}{\partial x^m},\quad m=0,1,2,\ldots$$

and

$$\Psi^{(0)}=\psi,\qquad \Psi^{(1)}=\psi_t=(1-\psi_x)F(x+\psi)=F_1(\Psi_0,\Psi_1)$$

take

$$\Psi^{(K)}=F_K(\Psi_0,\Psi_1,\ldots,\Psi_K)$$

then

$$\Psi^{(K+1)}=\sum_{l=0}^{K}\frac{\partial F_K}{\partial\Psi_l}\frac{\partial^l\Psi^{(1)}}{\partial x^l}=F_{K+1}(\Psi_0,\Psi_1,\ldots,\Psi_{K+1}).$$

Therefore, we obtain

$$\psi_K=\Psi^{(K)}|_{\Psi_0=0,\Psi_1=0,\ldots,\Psi_K=0}(x^*),\quad K=1,2,\ldots.$$

The approximation scheme is constructed by truncating series (44). When system (1) is an autonomous Hamiltonian system, because the expansion of the solution is unique, the scheme is sympletic. This expansion is new, as we know. The scheme constructed by truncating series (44) is called the generalized time-centered Euler scheme, which is the important application of Feng's theory of constructing symplectic schemes via the generating function.

3. THE CONSERVATION LAWS

The solution of system (1) is generated by the vector generating function $\psi(x,\bar{x})$:

$$\bar{x}=x+\sum_{K=1}^{+\infty}\psi_K(x,\bar{x})t^K. \tag{45}$$

If equation (45) has the linear conservation laws

$$C^{\mathrm{T}}\bar{x}=C^{\mathrm{T}}x \tag{46}$$

where

$$C=\begin{bmatrix}C_1\\C_2\\\vdots\\C_N\end{bmatrix}\in\mathbb{R}^N$$

then, we obtain

$$C^{\mathrm{T}}\psi_K(x,\bar{x})=0,\quad K=1,2,\ldots,\quad\forall x,\ \bar{x}\in\mathbb{R}^N. \tag{47}$$

The approximate scheme

$$x^{n+1}=x^n+\sum_{K=1}^{m}\psi_K(x^n,x^{n+1})t^K \tag{48}$$

has the same linear conservative quantities as system (1).

Now, we study the quadratic conservation laws for the generalized time-centered Taylor series

$$\bar{x} = x + \sum_{K=1}^{+\infty} \psi_K(\tfrac{1}{2}(x + \bar{x}))t^K/K!. \tag{49}$$

If equation (49) has the quadratic conservation laws, namely, there exists an $N \times N$ symmetric matrix A, such that

$$\bar{x}^{\mathrm{T}} A x = x^{\mathrm{T}} A x \tag{50}$$

then

$$\tfrac{1}{2}(\bar{x} + x)^{\mathrm{T}} \psi_K(\tfrac{1}{2}(\bar{x} + x)) = 0, \quad \forall K = 1, 2, \ldots, \quad \forall \bar{x},\ x \in \mathbb{R}^N. \tag{51}$$

Therefore, the scheme constructed by truncating expression (49)

$$x^{n+1} = x^n + \sum_{K=1}^{m} \psi_K(\tfrac{1}{2}(x^n + x^{n+1}))t^K/K! \tag{52}$$

has the same conservation laws as system (1). These properties have been mentioned previously [7] for general Hamiltonian systems. The scheme preserving the positive quadratic conservative quantity has nonlinear stability for long-time computation. Hence, the generalized time-centered Euler scheme can play an important role in the numerical time integration of semidiscrete systems of PDEs which have the quadratic conservation laws.

4. THE STABILITY OF THE GENERALIZED TIME-CENTERED EULER SCHEME

It is well-known that the scheme constructed by truncating the Taylor series is A-stable. Now we discuss the stability of the generalized time-centered Euler scheme. The author has previously given a detailed analysis on the L-stability of the time-centered Euler scheme for Hamiltonian systems [8]. The discussion given here about the L-stability of the generalized time-centered Euler scheme for general ODEs is different from that in Ref. [8]. The L-stability of difference schemes for Hamiltonian systems is a special problem.

We consider the linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Hx \tag{53}$$

the solution of the Hamiltonian–Jacobi equation (36) can be solved exactly:

$$\phi(W, t) = \tfrac{1}{2} W^{\mathrm{T}}\left(2J \tanh\left(\frac{\tau}{2} L\right)\right) W \tag{54}$$

$$L = J^{-1} S, \qquad S^{\mathrm{T}} = S \quad (2N \times 2N \text{ matrix}) \tag{55}$$

and

$$S = \begin{pmatrix} 0 & H^{\mathrm{T}} \\ H & 0 \end{pmatrix} \tag{56}$$

where H is an $N \times N$ matrix and

$$\begin{aligned} \tanh \lambda &= \lambda - \tfrac{1}{3}\lambda^3 + \tfrac{7}{15}\lambda^5 - \tfrac{17}{312}\lambda^7 + \cdots \\ &= \sum_{K=1}^{\infty} \alpha_{2K-1} \lambda^{2K-1} \end{aligned} \tag{57}$$

and

$$\alpha_{2K-1} = 2^{2K}(2^{2K} - 1)\mathrm{B}_{2K}/(2K)!;$$

B_{2K} are Bernoulli numbers.

Now the scheme can be rewritten as

$$\bar{x} - x = \sum_{K=1}^{+\infty} \alpha_{2K-1}\left(\frac{\tau}{2}\right)^{2K-1} H^{2K-1}\tfrac{1}{2}(\bar{x} + x). \tag{58}$$

Definition 1

The mapping A: $x \rightarrow \bar{x}$

$$\bar{x} = Ax$$

is L-stable (Liapunov stable) at x_0 if $\forall \epsilon > 0$, $\exists \delta > 0$, such that

$$|A^n x - A^n x_0| < 0, \quad \forall n,\ 0 \leqslant n < \infty$$

when $|x - x_0| < \delta$.

We construct a scheme for a linear autonomous system by truncating expression (57):

$$\bar{x} - x = \sum_{K=1}^{m} \alpha_{2K-1}\left(\frac{\tau}{2}\right)^{2K-1} H^{2K-1}\tfrac{1}{2}(\bar{x} + x). \tag{59}$$

Conclusion 1

If there exist positive matrices A and K, such that

$$H^T A + AH = -K \tag{60}$$

then, for sufficiently small t, the scheme is L-stable.

Proof. Define the A-inner product

$$(x, x)_A = x^{\mathrm{T}} A x, \quad \forall x \in \mathbb{R}^N$$

and the A-norm

$$\| x \|_A = \sqrt{(x, x)_A}.$$

Now we can define a norm for the matrix T: $\mathbb{R}^N \rightarrow \mathbb{R}^N$

$$\| A \|_A = \max_{x \in \mathbb{R}^N} \frac{\| Tx \|_A}{\| x \|_A}.$$

Let

$$M = \frac{1}{2} \sum_{K=1}^{m} \alpha_{2K-1}\left(\frac{\tau}{2}\right)^{2K-1} H^{2K-1}$$

under condition (60), we can easily prove, for sufficiently small t, that

$$y^{\mathrm{T}}(M^{\mathrm{T}} A + AM) y \leqslant 0, \quad \forall y \in \mathbb{R}^N.$$

Let

$$T = (I - M)^{-1}(I + M)$$

then

$$\| T \|_A = \max_{x \in \mathbb{R}^N} \frac{x^{\mathrm{T}}[(I + M)(I - M)^{-1}]^{\mathrm{T}} A (I + M)(I - M)^{-1} x}{x^{\mathrm{T}} A x}$$

$$= \max_{y \in \mathbb{R}^N} \frac{y^{\mathrm{T}} A y + y^{\mathrm{T}} M^{\mathrm{T}} A M y + y^{\mathrm{T}}(M^{\mathrm{T}} A + AM) y}{y^{\mathrm{T}} A y + y^{\mathrm{T}} M^{\mathrm{T}} A M y - y^{\mathrm{T}}(M^{\mathrm{T}} A + AM) y} \leqslant 1.$$

Therefore,

$$\| \bar{x} \|_A \leqslant \| x \|_A$$

hence, the scheme is L-stable.

Remark 1. The assumption of Conclusion 1 above can be weakened as follows, if there exists a positive matrix A and a nonnegative matrix K, such that

$$H^{\mathrm{T}}A + AH = -K \tag{60'}$$

then scheme (59) is L-stable.

Conclusion 2

Scheme (59) is L-stable iff $\operatorname{Re}\lambda < 0$ or $\operatorname{Re}\lambda = 0$ and λ has the simple elementary divisors, where λ is an eigenvalue of H.

From Conclusion 2, we obtain the following.

Conclusion 3

The generalized time-centered Euler scheme is L-stable.

5. DISCUSSION

Recently, the symplectic approximation to the noncanonical Hamiltonian system

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = J(Z)H_Z \tag{61}$$

has been studied by Dr Daoliu Wang (unpublished). He considered some special cases of equation (61) and devised the symplectic schemes. The extension of the time-centered Euler scheme provides a means to study the approximation to noncanonical Hamiltonian systems. Because the expansion of the solution of equation (61) at the time-centered point

$$\tfrac{1}{2}(\bar{Z} + Z) \tag{62}$$

is unique, if the symplectic approximation to equation (61) is a time-centered one-step scheme, the scheme can be obtained by our method. When $J(Z)$ is a constant matrix, the author [7] has systematically studied the symplectic approximation to it. The application of this type of scheme will be presented in a future publication.

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